Function Sketching Techniques

The graphs of many simple functions of the form \( y = \frac{g(x)}{h(x)} \) are easily obtained by considering the following key properties:

1. **Factor** \( g(x) \) and \( h(x) \) if they are polynomials.

2. **Intercept on the y-axis**

\[ x = 0 \quad \Rightarrow \quad y = \frac{g(0)}{h(0)} \]

*Note: since all functions are single valued there will only be one intercept on the y-axis.*

3. **Intercept on the x-axis**

\[ g(x) = 0 \quad \Rightarrow \quad x = ?, ?, \ldots \]

*Note: there may be more than one intercept on the x-axis. The intercepts occur at the zeros of \( g(x) \). These points are most easily found from the factored form of \( g(x) \).*

4. **Vertical Asymptotes**

\[ h(x) = 0 \quad \Rightarrow \quad x = ?, ?, ?, \ldots \]

*Note: vertical asymptotes are found at the zeros of the bottom function \( h(x) \).*
(5) **Horizontal or Slant Asymptotes**

\[ y \to ? \quad \text{as} \quad x \to \pm \infty \]

**Note** Horizontal or slant asymptotes are found by examining the behaviour of \( y = f(x) \) as \( x \to \pm \infty \).

(6) **Sign of the Function**

**Note**
1. Find the sign of \( y \) at \( x = \pm \infty \) or \( -\infty \).
2. The sign of the function can change at the zeros of \( g(x) \) and \( h(x) \).
3. At **odd** power factors \( \Rightarrow \) sign changes
   
   **even** power factors \( \Rightarrow \) sign unchanged

(7) **Symmetry**

**Even Function** \( y(-x) = y(x) \) Symmetry about the \( x \)-axis

**Odd Function** \( y(-x) = -y(x) \) Symmetry about the origin.

**Function Sketching Examples**

**Example 1** Sketch the graph of

\[ y = \frac{x^2 + 2x + 1}{x - 2} \]

\[ = \frac{(x+1)^2}{(x-2)} \]
Example 2

\[ y = \frac{x^2 + x - 2}{2x^2 - 4x - 6} \]

\[ = \frac{(x+2)(x-1)}{2(x-3)(x+1)} \]

\[ x = 0 \Rightarrow y = \frac{1}{3} \]
\[ y = 0 \Rightarrow x = -2, 1 \]
\[ y \to \infty \text{ as } x \to 3, -1 \]
\[ y \to \frac{1}{2} \text{ as } x \to \pm \infty \]