The Impact of Market Clearing Assumptions and Dynamics on Demand Elasticities

Proof of Equation (6)

We will prove the following result for the equilibrium estimator of the model defined by equations (1) - (4).

\[
p \lim(\hat{\alpha}_1 - \alpha_1) = \frac{p \lim \{(I - M_{2e}^*)P\}'g\}}{\gamma \cdot p \lim \{P'M_1\hat{P}\}}
\]  
\[
\text{where } M_{2e}^* = I - x_{2e}^*(x_{2e}^*x_{2e}^*)^{-1}x_{2e}^*, \quad x_{2e}^* = M_1x_{2e}, \quad M_1 = I - x_1(x_1'x_1)^{-1}x_1'
\]

We assume that equation (5) is the data generating process but we erroneously omit \( \hat{g} \) from (5) and therefore estimate the following via OLS:

\[
Q = x_1\beta_1 + \alpha_1\hat{P} + u
\]

Using partitioned regression (Greene, 1993, p179), the OLS estimator of \( \alpha_1 \) in (7) can be written as:

\[
\hat{\alpha}_1 = (\hat{P}'M_1\hat{P})^{-1}\hat{P}'M_1(x_1\beta_1 + \alpha_1\hat{P} + (1/\gamma)\hat{g} + u_1)
\]

where we have replaced \( Q \) by (5). Expanding (8) gives:

\[
\hat{\alpha}_1 = (\hat{P}'M_1\hat{P})^{-1}\hat{P}'M_1x_1\beta_1 + (\hat{P}'M_1\hat{P})^{-1}\hat{P}'M_1\alpha_1\hat{P} + (\hat{P}'M_1\hat{P})^{-1}\hat{P}'M_1(1/\gamma)\hat{g} + (\hat{P}'M_1\hat{P})^{-1}\hat{P}'M_1u_1
\]

Since \( M_1x_1 = 0 \) the first term disappears, the second term simplifies and we get:

\[
\hat{\alpha}_1 = \alpha_1 + (1/\gamma)(\hat{P}'M_1\hat{P})^{-1}\hat{P}'M_1\hat{g} + (\hat{P}'M_1\hat{P})^{-1}\hat{P}'M_1u_1
\]

Taking the plim of both sides we get:

\[
p \lim(\hat{\alpha}_1) = \alpha_1 + (1/\gamma)\frac{p \lim \{\hat{P}'M_1\hat{g}\}}{p \lim \{\hat{P}'M_1\hat{P}\}} + \frac{p \lim \{\hat{P}'M_1u_1 / n\}}{p \lim \{\hat{P}'M_1\hat{P} / n\}}
\]

We assume \( p \lim \{M_1u_1 / n\} = 0 \) and thus the last term disappears and we can write:

\[
p \lim(\hat{\alpha}_1 - \alpha_1) = \frac{p \lim \{\hat{P}'M_1\hat{g}\}}{\gamma \cdot p \lim \{\hat{P}'M_1\hat{P}\}}
\]

(9)
Focus upon the numerator of (9):

\[
\hat{P}'M_1g = \{(I - M)P\}'M_1(I - M)g = P'{(I - M)'M_1(I - M)}g \tag{10}
\]

Consider the middle term in (10), since \((I - M)\) is idempotent and expanding we get:

\[
(I - M)'M_1(I - M) = M_1 - M_1M - MM_1 + MM_1M
\]

Since \(MM_1 = M\) and \(M\) is idempotent we get:

\[
(I - M)'M_1(I - M) = M_1 - M_1M = M_1(I - M) \tag{11}
\]

Substitute (11) into (10);

\[
\hat{P}'M_1g = P'M_1(I - M)g = P'M_1\{x_1\hat{\beta}_{g1} + x_{2e}\hat{\beta}_{g2}\} \tag{12}
\]

where \(\hat{\beta}_{g1}\) are the regression coefficients gained by regressing \(g\) on \(x_1\) and \(x_{2e}\). Since \(M_1x_1 = 0\), (12) becomes:

\[
\hat{P}'M_1g = P'M_1x_{2e}\hat{\beta}_{g2} \tag{13}
\]

\(\hat{\beta}_{g2}\) can be determined using partitioned regression concepts:

\[
\hat{\beta}_{g2} = (x_{2e}'M_1x_{2e})^{-1}x_{2e}'M_1g \tag{14}
\]

Substitute (14) into (13) and simplify recognising that \(M_1\) is idempotent:

\[
\hat{P}'M_1g = P'x_{2e}^* (x_{2e}^* x_{2e}^*)^{-1}x_{2e}^* g = P'(I - M_{2e}^*)g \tag{15}
\]

Substitute (15) into (9) and we get our result:

\[
p \lim(\hat{\alpha}_1 - \alpha_1) = \frac{p \lim \{[(I - M_{2e}^*)P]'g\}}{\gamma \cdot p \lim \{\hat{P}'M_1\hat{P}\}}
\]