Applications of Functions

Example 1

Average Monthly Temperature in Sydney

<table>
<thead>
<tr>
<th>Month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature</td>
<td>18.6</td>
<td>19</td>
<td>18.1</td>
<td>15</td>
<td>11.7</td>
<td>9.4</td>
<td>8</td>
<td>9.3</td>
<td>11.6</td>
<td>14</td>
<td>15.7</td>
<td>17.5</td>
</tr>
</tbody>
</table>

\[ F(t) = 13.5 + 5.5\cos\left(\frac{\pi(t-2)}{6}\right) \]
Example 2 The area $A$ of a circle of radius $r$ is given by

$$A = \pi r^2$$

Example 3 You have just bought a brand new car and plan to drive it from Wagga to Sydney. The odometer reads 100km when you get in the car. You plan to drive around 100km/h. Find or estimate:

(1) The odometer reading after you have driven for 2.5 hours.

(2) The time it will take until your odometer reaches 300km.

<table>
<thead>
<tr>
<th>$t$ (hours)</th>
<th>0.0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$ (km)</td>
<td>100</td>
<td>200</td>
<td>300</td>
<td>400</td>
</tr>
</tbody>
</table>

$S = 100t + 100$
Linear Functions

Slope Intercept Form

\[ y = mx + b \]

\[ m = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{h}{k} \]

\[ b = \text{Vertical Intercept} \]

Point-Slope Equation

\[ y - y_0 = m(x - x_0) \]

\((x_0, y_0) = \text{A point on the Line}\)

\(m = \text{Slope of the Line}\)

Two Point Equation

\[ \frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0} \]

\((x_0, y_0) \text{ and } (x_1, y_1) = \text{Are two points on the Line}\)
Limit Theorems (0)

Limit of a Function

\[ \lim_{x \to a} F(x) = L \]

Examples

\[ \lim_{x \to 1} \frac{x^2 + 1}{2x + 1} = \frac{2}{3} \]

\[ \lim_{x \to 0} \frac{\sin x}{x} = 1 \]

Definition of the Limit (after Karl Weierstrass 1815 - 1897)

The number \( L \) is the limit of \( F(x) \) as \( x \to a \), provided that, given any number \( \varepsilon > 0 \), there exists a number \( \delta > 0 \) such that

\[ |F(x) - L| < \varepsilon \]

For all \( x \) such that

\[ 0 < |x - a| < \delta \]
Limit Theorems (1)

Combination Theorem

If \(\lim_{x \to a} F_1(x) = L_1\) and \(\lim_{x \to a} F_2(x) = L_2\) then

1. \(\lim_{x \to a} (F_1(x) + F_2(x)) = L_1 + L_2\)
2. \(\lim_{x \to a} (F_1(x) - F_2(x)) = L_1 - L_2\)
3. \(\lim_{x \to a} (F_1(x) \cdot F_2(x)) = L_1 \cdot L_2\)
4. \(\lim_{x \to a} (k \cdot F_1(x)) = kL_1\)
5. \(\lim_{x \to a} \left(\frac{F_1(x)}{F_2(x)}\right) = \frac{L_1}{L_2}\)

Composition Theorem

Suppose that \(\lim_{x \to a} g(x) = L\) and that \(\lim_{x \to L} f(x) = f(L)\)

Then \(\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x) = f(L)\)
Limit Theorems (2)

Sandwich Theorem

Suppose that \( f(x) \leq g(x) \leq h(x) \) for all \( x \neq a \) in some neighborhood of \( a \) and also that

\[
\lim_{{x \to a}} f(x) = L = \lim_{{x \to a}} h(x)
\]

then \( \lim_{{x \to a}} g(x) = L \)
Continuity of a Function

A function \( y = f(x) \) is continuous at a point \( x = a \) \textit{if and only if} all three of the following statements are true.

1. \( f(a) \) exists
2. \( \lim_{x \to a} f(x) \) exists
3. \( \lim_{x \to a} f(x) = f(a) \)

\[ \text{Fig. 2.4.2 A graph that is not continuous} \]

\[ \text{Graph showing discontinuity at } x = a \]
The Achilles

The Second Paradox of Zeno (494 - 435 BC)

"Archilles running to overtake a crawling tortoise ahead of him can never overtake it, because he must first reach the place from which the tortoise started; when Achilles reaches that place, the tortoise has departed and so is still ahead. Repeating the argument we easily see that the tortoise will always be ahead"
Function Sketching Techniques

\[ y = f(x) = \frac{g(x)}{h(x)} \]

(1) Factor \( g(x) \) and \( h(x) \) if they are polynomials

(2) Intercept on the y axis

\[ x = 0 \quad \Rightarrow \quad y = f(0) = \frac{g(0)}{h(0)} \]

(3) Intercepts on the x axis

\[ g(x) = 0 \quad \Rightarrow \quad x = ?, ?, ? \]

(4) Vertical Asymptotes

\[ h(x) = 0 \quad \Rightarrow \quad x = ?, ?, ? \]

(5) Horizontal or Skew Asymptotes

\[ y \to ? \quad \text{as} \quad x \to \pm\infty \]

(6) Sign of the Function

Odd Power Factors : Sign Changes
Even Power Factors : Sign Unchanged

(7) Symmetry

Even Function : x-axis \( f(-x) = f(x) \)
Odd Function : origin \( f(-x) = -f(x) \)

(8) Location & nature of stationary points

(i) Maximum \( f'(x) = 0 \) and \( f''(x) < 0 \)
(ii) Minimum \( f'(x) = 0 \) and \( f''(x) > 0 \)
(iii) Inflection \( f''(x) = 0 \)
Table of Derivatives

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$D f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2$</td>
<td>$2x^1$</td>
</tr>
<tr>
<td>$x^n$</td>
<td>$nx^{n-1}$</td>
</tr>
<tr>
<td>$\ln x$</td>
<td>$\frac{1}{x}$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x$</td>
</tr>
<tr>
<td>$\sin x$</td>
<td>$\cos x$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$-\sin x$</td>
</tr>
<tr>
<td>$\tan x$</td>
<td>$\sec^2 x$</td>
</tr>
<tr>
<td>$\cot x$</td>
<td>$-\csc^2 x$</td>
</tr>
<tr>
<td>$\sec x$</td>
<td>$\sec x \tan x$</td>
</tr>
<tr>
<td>$\csc x$</td>
<td>$-\csc x \cot x$</td>
</tr>
<tr>
<td>$\sin^{-1} x$</td>
<td>$\frac{1}{\sqrt{1 - x^2}}$</td>
</tr>
<tr>
<td>$\tan^{-1} x$</td>
<td>$\frac{1}{1 + x^2}$</td>
</tr>
</tbody>
</table>
# Table of Integrals

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$\int f(x) , dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$\frac{x^{n+1}}{n+1} + c \quad n \neq -1$</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x + c$</td>
</tr>
<tr>
<td>$\sin x$</td>
<td>$-\cos x + c$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$\sin x + c$</td>
</tr>
<tr>
<td>$\sec^2 x$</td>
<td>$\tan x + c$</td>
</tr>
<tr>
<td>$\csc^2 x$</td>
<td>$-\cot x + c$</td>
</tr>
<tr>
<td>$\sec x \tan x$</td>
<td>$\sec x + c$</td>
</tr>
<tr>
<td>$\csc x \cot x$</td>
<td>$-\csc x + c$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{a^2 - x^2}}$</td>
<td>$\sin^{-1}\left(\frac{x}{a}\right) + c$</td>
</tr>
<tr>
<td>$\frac{1}{a^2 + x^2}$</td>
<td>$\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$</td>
</tr>
</tbody>
</table>
Derivative Rules

Addition Rule
\[
\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)
\]

Constant Rule
\[
\frac{d}{dx} [C f(x)] = C \frac{d}{dx} f(x)
\]

Product Rule
\[
\frac{d}{dx} [f(x)g(x)] = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)
\]

Quotient Rule
\[
\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{[g(x)]^2}
\]

Chain Rule
\[
\frac{d}{dx} \{f(g(x))\} = f'(g(x)) \frac{d}{dx} g(x)
\]
Derivative Example - Rates of Change

The free-fall distance covered by a base-jumper is given by,

\[ D(t) = 5t^2 \]

<table>
<thead>
<tr>
<th>( t ) (sec)</th>
<th>( D ) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>45</td>
</tr>
<tr>
<td>4</td>
<td>80</td>
</tr>
</tbody>
</table>

Where \( t \) is time in seconds and \( D \) is distance in metres.

Estimate the velocity of the base-jumper 2 seconds after the jump.

\[
\text{Average Velocity} = \frac{\text{Distance Covered}}{\text{Time Interval}} = \frac{\Delta D}{\Delta t} = \frac{45 - 20}{3 - 2} = 25 \text{ m/s}
\]

Graph of Distance vs Time
Definition of the Derivative

The derivative of a function \( f \) is the function \( f' \), whose value is defined by the equation,

\[
f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

whenever the limit exists.

**Geometrically**, the derivative is the slope of the tangent to \( f(x) \) at \( x \), which is calculated as the limiting slope of the chord between \( x \) and \( x + \Delta x \).
Trigonometric Functions (1)

Basic definitions

\[
\begin{align*}
\cos(\theta) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\
\sin(\theta) &= \frac{\text{opposite}}{\text{hypotenuse}} \\
\tan(\theta) &= \frac{\text{opposite}}{\text{adjacent}}
\end{align*}
\]

Radian Measure

\[
\begin{align*}
360^\circ &= 2\pi \text{ radians} & 90^\circ &= \frac{\pi}{2} \text{ radians} & 45^\circ &= \frac{\pi}{4} \text{ radians} \\
180^\circ &= \pi \text{ radians} & 60^\circ &= \frac{\pi}{3} \text{ radians} & 30^\circ &= \frac{\pi}{6} \text{ radians}
\end{align*}
\]

ARC Length & Radian Measure

\[
\begin{align*}
s &= r\theta \\
\theta &= \frac{s}{r} \text{ radians}
\end{align*}
\]
Trigonometric Functions (2)

Important Trigonometric Values

\[
\begin{align*}
\sin \frac{\pi}{3} &= \frac{\sqrt{3}}{2} & \sin \frac{\pi}{6} &= \frac{1}{2} \\
\cos \frac{\pi}{3} &= \frac{1}{2} & \cos \frac{\pi}{6} &= \frac{\sqrt{3}}{2} \\
\tan \frac{\pi}{3} &= \sqrt{3} & \tan \frac{\pi}{6} &= \frac{1}{\sqrt{3}}
\end{align*}
\]

\[
\begin{align*}
\sin \frac{\pi}{4} &= \frac{1}{\sqrt{2}} \\
\cos \frac{\pi}{4} &= \frac{1}{\sqrt{2}} \\
\tan \frac{\pi}{4} &= 1
\end{align*}
\]

\[
\begin{align*}
\sin(0) &= 0 & \sin\left(\frac{\pi}{2}\right) &= 1 & \sin(\pi) &= 0 \\
\cos(0) &= 1 & \cos\left(\frac{\pi}{2}\right) &= 0 & \cos(\pi) &= -1
\end{align*}
\]

Other Trigonometric Functions

\[
\begin{align*}
cosec \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta}
\end{align*}
\]
Trigonometric Identities

Pythagorean Formulas

\[
\sin^2 \theta + \cos^2 \theta = 1 \quad *
\]
\[
\tan^2 \theta + 1 = \sec^2 \theta \quad : \text{Divide by } \cos^2 \theta
\]
\[
1 + \cot^2 \theta = \csc^2 \theta \quad : \text{Divide by } \sin^2 \theta
\]

Double Angle Formulas

\[
\sin 2\theta = 2 \sin \theta \cos \theta \quad *
\]
\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta \quad * \implies
\]
\[
\sin^2 \theta = \frac{1 - \cos 2\theta}{2}
\]
\[
\cos^2 \theta = \frac{1 + \cos 2\theta}{2}
\]

Sum & Difference Formulas

\[
\sin(A + B) = \sin A \cos B + \cos A \sin B
\]
\[
\sin(A - B) = \sin A \cos B - \cos A \sin B
\]
\[
\cos(A + B) = \cos A \cos B - \sin A \sin B
\]
\[
\cos(A - B) = \cos A \cos B + \sin A \sin B
\]

Two Important Limits

\[
\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1
\]
\[
\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0
\]
Modelling Periodic Data

\[ F(t) = M + A \cos \left( \frac{2\pi}{P} (t + \varphi) \right) \]

where

- \( M \) = Mean Signal Level
- \( A \) = Amplitude of the Oscillation
- \( P \) = Period of the Oscillation
- \( \varphi \) = Phase Difference

Example: Monthly Sydney Temperature

<table>
<thead>
<tr>
<th>Month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp (C)</td>
<td>18.6</td>
<td>19.0</td>
<td>18.1</td>
<td>15.0</td>
<td>11.7</td>
<td>9.4</td>
<td>8.0</td>
<td>9.3</td>
<td>11.6</td>
<td>14.0</td>
<td>15.7</td>
<td>17.5</td>
</tr>
</tbody>
</table>

Maximum Temperature = 19
Minimum Temperature = 8
Amplitude \( A \) = \((19-8)/2 = 5.5\)
Period \( P \) = 12
Phase \( \varphi \) = -2
Mean \( M \) = \((19+8)/2 = 13.5\)
Sydney Temperature Model

\[ F(t) = 13.5 + 5.5 \cos \left[ \frac{2\pi}{12} (t - 2) \right] \]
\[ = 13.5 + 5.5 \cos \left[ \frac{\pi}{6} (t - 2) \right] \]

Sydney Temperature Model

Rate of Sydney Temperature Change in April

\[ \frac{dT}{dt} = -5.5 \sin \left[ \frac{\pi}{6} (t - 2) \right] \frac{d}{dt} \left[ \frac{\pi}{6} (t - 2) \right] \]
\[ = -5.5 \frac{\pi}{6} \sin \left[ \frac{\pi}{6} (t - 2) \right] \]
\[ = -2.5 \text{ (C)/Month} \]
Laws of Exponents & Logarithms

\[ a^m \times a^n = a^{m+n} \]
\[ a^m \div a^n = \frac{a^m}{a^n} = a^{m-n} \]
\[ (a^m)^n = a^{mn} \]
\[ a^0 = 1 \]
\[ a^{-1} = \frac{1}{a} \]
\[ a^n = \sqrt[n]{a} \]

If \( a^x = y \) \( x = \log_a y \)
\( \log_a a^x = x \) \( a^{\log_a y} = y \)

\[ \log_a PQ = \log_a P + \log_a Q \]
\[ \log_a \frac{P}{Q} = \log_a P - \log_a Q \]
\[ \log_a P^n = n \log_a P \]
\[ \log_a 1 = 0 \]
\[ \log_b x = \frac{\log_a x}{\log_a b} \]
Exponential Functions

Basic Property of Exponential Functions

“The same DIFFERENCE between two $x$-values always corresponds to the same RATIO between the corresponding $y$-values”
Exponential Functions – Population Growth

\[ P(t) = P_0 e^{kt} \]

Where \( P_0 \) = initial population at time \( t = 0 \)
\( k \) = growth rate constant.

Note: This growth model has the property that

\[ \frac{dP}{dt} = kP_0 e^{kt} = kP \]


<table>
<thead>
<tr>
<th>Year (y)</th>
<th>Time (t - 1980)</th>
<th>Population (P) mil</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980</td>
<td>0</td>
<td>67.38</td>
</tr>
<tr>
<td>1981</td>
<td>1</td>
<td>69.13</td>
</tr>
<tr>
<td>1982</td>
<td>2</td>
<td>70.93</td>
</tr>
<tr>
<td>1983</td>
<td>3</td>
<td>72.77</td>
</tr>
<tr>
<td>1984</td>
<td>4</td>
<td>74.66</td>
</tr>
<tr>
<td>1985</td>
<td>5</td>
<td>76.60</td>
</tr>
<tr>
<td>1986</td>
<td>6</td>
<td>78.59</td>
</tr>
</tbody>
</table>

\[ P(0) = 67.38 = P_0 e^{0k} \]
\[ P(3) = 72.77 = P_0 e^{3k} \]
\[ e^{3k} = \frac{72.77}{67.38} \]
\[ k = \frac{1}{3} \ln \left( \frac{72.77}{67.38} \right) = 0.0257 \]
Exponential Functions – Population Growth

Population Model of Mexico (1980-1986)

\[ P(t) = 67.38e^{0.0257t} \]

Population prediction for 1986

1986 \hspace{1cm} t = 6 \hspace{1cm} P(6) = 67.38e^{0.0257 \times 6} = 78.61 \text{ Mil}

Note: The actual population of Mexico in 1986 was 78.59 Mil
Exponential Decay

Drug Elimination

The amount $A(t)$ of a drug in the human bloodstream, in excess of the natural level, declines at a rate proportional to that excess amount present. That is;

$$\frac{dA(t)}{dt} = -\lambda A(t)$$

so that

$$A(t) = a_0 e^{-\lambda t}$$

The parameter $\lambda$ is called the elimination constant, and $\frac{1}{\lambda}$ is called the elimination time.

Example - Alcohol: If the elimination time for alcohol in a typical person is $T = \frac{1}{\lambda} = 2.5 \text{ hr}$, how long will it take for that person’s blood alcohol level to drop from 0.08% to 0.05%?

Assuming that the normal concentration of alcohol in the blood is zero then;

$$A(t) = A_0 e^{-\lambda t}$$

$$0.05 = 0.08e^{-\frac{1}{2.5}t}$$

$$-0.4t = \ln\left(\frac{0.05}{0.08}\right)$$

$$t = 1.2 \text{ hr}$$
Stationary Points

The stationary points for a function include relative maxima, relative minima and points of inflection.

- **Maximum** \( f''(x) = 0 \) and \( f'''(x) < 0 \)
- **Minimum** \( f''(x) = 0 \) and \( f'''(x) > 0 \)
- **Inflection** \( f''(x) = 0 \)

**Example**
Find all stationary points of the function \( f(x) = x^3 - 2x^2 \)

\[
f''(x) = 3x - 4\]
\[
f'''(x) = 6x - 4\]

**Analysis**

\[
f(x) = x^3 - 2x^2 \quad f'(x) = 3x^2 - 4x \quad f''(x) = 6x - 4\]
\[
f'(x) = 3x(x - \frac{4}{3}) = 0 \quad \Rightarrow \quad x = 0 \quad f'''(0) = -4 \quad \text{Maximum}\]
\[
\quad \Rightarrow \quad x = \frac{4}{3} \quad f''(\frac{4}{3}) = 4 \quad \text{Minimum}\]
\[
f''(x) = 6x - 4 = 0 \quad \Rightarrow \quad x = \frac{2}{3} \quad \text{Inflection}\]
Symmetry

Symmetry is a useful functional property to examine when sketching many graphs.

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Reflection</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even</td>
<td>y-axis</td>
<td>$f(-x) = f(x)$</td>
</tr>
<tr>
<td>Odd</td>
<td>Origin</td>
<td>$f(-x) = -f(x)$</td>
</tr>
</tbody>
</table>

**Symmetry Examples**

- $f(x) = x^2$ \quad $f(-x) = (-x)^2 = x^2 = f(x)$ \quad Even
- $f(x) = \sin x$ \quad $f(-x) = \sin(-x) = -\sin x = -f(x)$ \quad Odd
- $f(x) = x^3 - 2x^2$ \quad $f(-x) = (-x)^3 - 2(-x)^2$  
  \hspace{1cm} $= -x^3 - 2x^2$
  \hspace{1cm} $\neq f(x)$
  \hspace{1cm} $\neq -f(x)$ \quad None
Newton’s Method

This is an iterative method for finding the roots of

\[ f(x) = 0 \]

based on tangent line approximations.

The equation of the tangent line at \( x_n \) is given by,

\[
\frac{y - f(x_n)}{x - x_n} = f'(x_n)
\]

If we take the next approximation to the root \( x_{n+1} \) as the point where the tangent line crosses the x-axis (\( y = 0 \)) we have Newton’s method for finding the next iterative approximation to the root as;

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]
Newton’s Method

Example: Finding Square Roots

Find $\sqrt{2}$ by using Newton’s method to solve for the roots of the equation:

$$f(x) = x^2 - 2 = 0$$

starting with the initial approximation $x_0 = 1$.

Newton’s iteration scheme is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - 2}{2x_n}$$

$$= \frac{x_n^2 + 2}{2x_n}$$

This generates the sequence of approximations

$$x_0 = 1.00000000$$
$$x_1 = 1.50000000$$
$$x_2 = 1.41666667$$
$$x_3 = 1.41421569$$
$$x_4 = 1.41421356$$
$$x_5 = 1.41421356$$

that is accurate to 8 decimal places after only 4 iterations.
Related Rates

Police Radar

Police radar is located behind a tree 40 m up a side lane and is monitoring a school zone 100 m further along the main road. A solicitor, booked at 43 km/hr, challenges the decision in court on the basis that his measured speed may have been too high due to the slant angle of the radar measurement. Is he likely to win his case?

Analysis:

\[ 40^2 + x(t)^2 = z(t)^2 \]

\[ \frac{d}{dt} \left( 40^2 + x(t)^2 = z(t)^2 \right) \]

\[ 0 + 2x \frac{dx}{dt} = 2z \frac{dz}{dt} \]

\[ \frac{dx}{dt} = \frac{z}{x} \frac{dz}{dt} = \frac{\sqrt{40^2 + 100^2}}{100} \cdot 43 = 46 \text{ km/hr} \]

Result: Solicitors true road speed is 46 km/hr
Useful Derivative Theorems (1)

Rolle’s Theorem

Suppose \( y = f(x) \) is continuous at every point of the closed interval \([a,b]\) and is differentiable at every point of its interior \((a,b)\). If \( f(a) = f(b) = 0 \), then there is at least one number \( c \) between \( a \) and \( b \) at which \( f'(c) = 0 \).
Useful Derivative Theorems (2)

Mean Value Theorem

If \( y = f(x) \) is continuous at every point of the closed interval \([a, b]\) and is differentiable at every point of its interior \((a, b)\). Then there is at least one number \( c \) between \( a \) and \( b \) at which

\[
\frac{f(b) - f(a)}{b - a} = f''(c)
\]
Useful Derivative Theorems (3)

L’Hopital’s Rule

Suppose that \( f(a) = g(a) = 0 \), and that \( f'(a) \) and \( g'(a) \) exist, and \( g'(a) \neq 0 \), then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}
\]

**Proof**  Consider

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}
\]

\[
= \lim_{x \to a} \frac{[f(x) - f(a)]/(x - a)}{[g(x) - g(a)]/(x - a)}
\]

\[
= \lim_{x \to a} \frac{[f(x) - f(a)]}{[g(x) - g(a)]} = \lim_{x \to a} \frac{f(x)}{g(x)} \quad \text{since} \quad f(a) = g(a) = 0
\]
Definite Integral - Reiman Integral

\[ \int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x \]

Upper Sum

\[ S_U = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x \]

\[ = \lim_{n \to \infty} \left[ \frac{1}{n} \left( \frac{1}{n} \right)^2 + \left( \frac{2}{n} \right)^2 + \cdots + \left( \frac{n}{n} \right)^2 \right] \]

\[ = \lim_{n \to \infty} \left[ \frac{1}{n^3} \left( 1^2 + 2^2 + \cdots + n^2 \right) \right] \]

\[ = \lim_{n \to \infty} \left[ \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \right] \]

\[ = \lim_{n \to \infty} \left[ \frac{(n+1)(2+\frac{1}{n})}{6} \right] \]

\[ = \frac{1}{3} \]

Lower Sum

\[ S_L = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x \]

\[ = \lim_{n \to \infty} \left[ \frac{1}{n} \left( 0^2 + \left( \frac{1}{n} \right)^2 + \cdots + \left( \frac{n-1}{n} \right)^2 \right) \right] \]

\[ = \lim_{n \to \infty} \left[ \frac{1}{n^3} \left( 2^2 + 1^2 + \cdots + (n-1)^2 \right) \right] \]

\[ = \lim_{n \to \infty} \left[ \frac{1}{n^3} \frac{(n-1)n(2n-1)}{6} \right] \]

\[ = \lim_{n \to \infty} \left[ \frac{(1-\frac{1}{n})(2-\frac{1}{n})}{6} \right] \]

\[ = \frac{1}{3} \]
First Fundamental Theorem of Integral Calculus

The Cumulative Area Function

\[ F_a(x) = \int_a^x f(t) \, dt \]

in an antiderivative of \( f(x) \).

Proof

\[
\frac{d}{dx} F_a(x) = \lim_{\Delta x \to 0} \frac{F_a(x + \Delta x) - F_a(x)}{\Delta x} \\
= \lim_{\Delta x \to 0} \left( \int_a^{x+\Delta x} f(t) \, dt - \int_a^x f(t) \, dt \right) / \Delta x \\
= \lim_{\Delta x \to 0} \frac{\Delta x f(x)}{\Delta x} \\
= f(x)
\]
Second Fundamental Theorem of Integral Calculus

If \( f(t) \) is continuous at every point of \([a, b]\), and \( F(t) \) is any antiderivative of \( f(t) \) on \([a, b]\), then

\[
\int_a^b f(t) \, dt = F(b) - F(a)
\]

\[= F(t)]_a^b \]

Proof

From the First Fundamental Theorem

\[ F_a (x) = \int_a^x f(t) \, dt \]

is an antiderivative of \( f(x) \). By the addition of an arbitrary constant \( C \) we can write another antiderivative \( F(x) \) as

\[ F(x) = \int_a^x f(t) \, dt + C \]

Letting \( x = a \) and remembering \( \int_a^a f(t) \, dt = 0 \) we obtain \( F(a) = C \), so that,

\[ F(x) = \int_a^x f(t) \, dt + F(a) \]

Now letting \( x = b \) we have

\[ \int_a^b f(t) \, dt = F(b) - F(a) \]
Sign of the Definite Integral

1. \( \int_{a}^{b} f(x) \, dx = + \text{Area} \parallel [f^+, \Delta x+] \)
2. \( \int_{b}^{a} f(x) \, dx = - \text{Area} \parallel [f^+, \Delta x-] \)
3. \( \int_{c}^{b} f(x) \, dx = - \text{Area} = [f^-, \Delta x+] \)
4. \( \int_{c}^{b} f(x) \, dx = + \text{Area} = [f^-, \Delta x-] \)
Properties of the Definite Integral

1. \( \int_{a}^{a} f(x) \,dx = 0 \)

2. \( \int_{a}^{b} f(x) \,dx = -\int_{a}^{a} f(x) \,dx \)

3. \( \int_{a}^{b} kf(x) \,dx = k \int_{a}^{b} f(x) \,dx \)

4. \( \int_{a}^{b} [f(x) + g(x)] \,dx = \int_{a}^{b} f(x) \,dx + \int_{a}^{b} g(x) \,dx \)

5. If \( f(x) \leq g(x) \) on \([a, b]\)
   then \( \int_{a}^{b} f(x) \,dx \leq \int_{a}^{b} g(x) \,dx \)

6. \( \int_{a}^{c} f(x) \,dx = \int_{a}^{b} f(x) \,dx + \int_{b}^{c} f(x) \,dx \)
Definite Integral - Signed Areas

\[ \int_a^b f(x) \, dx = \text{Signed Area under } f(x) \]

Fundamental Theorem

\[ \int_a^b f(x) \, dx = F(b) - F(a) \]

Where \( F(x) \) is an antiderivative or indefinite integral of \( f(x) \), such that

\[ \frac{d}{dx} F(x) = f(x) \]
First Fundamental Theorem

The Cumulative Area Function

\[ F_a(x) = \int_a^x f(t) \, dt \]

is an Antiderivative

\[ \frac{d}{dx} F_a(x) = \lim_{\Delta x \to 0} \frac{F_a(x + \Delta x) - F_a(x)}{\Delta x} \]

\[ = \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t) \, dt - \int_a^x f(t) \, dt}{\Delta x} \]

\[ = \lim_{\Delta x \to 0} \frac{f(x) \Delta x}{\Delta x} \]

\[ = f(x) \]
Methods of Integration

Simple Integration Rules

\[ \int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx \]

\[ \int C \, f(x) \, dx = C \int f(x) \, dx \]

\[ \int \frac{d}{dx} f(x) \, dx = \frac{d}{dx} \int f(x) \, dx = f(x) \]

Integration by Parts

\[ \text{D I } = ( ) \text{ I } - \int \text{ I D } \, dx \quad \text{Mnemonic} \]

\[ \int f(x)g(x) \, dx = f(x) \left[ \int g(x) \, dx \right] - \int \left\{ \left[ \int g(x) \, dx \right] \times \left[ \frac{d}{dx} f(x) \right] \right\} \, dx \]

Chain Rule Backwards

\[ \int f(g(x)) \frac{d}{dx} g(x) \, dx = F(g(x)) \]

where \( F \) is an antiderivative of \( f \)
Trapezoidal Rule Integration

Elementary Trapezoidal Rule

\[ \int_{0}^{h} f(x) \, dx = \frac{1}{2} [Base \times Sum \parallel Sides] \]

\[ = \frac{h}{2} [f(0) + f(h)] \]

Compound Trapezoidal Rule

\[ \int_{a}^{b} f(x) \, dx = \frac{h}{2} \left[ f(a) + 2f(a+h) + \cdots + 2f(b-h) + f(b) \right] \]

\[ = \frac{h}{2} \sum w \times f \]

\[ h = \frac{b-a}{n} \]
Simpson Rule Integration

Elementary Simpson Rule

\[ \int_0^{2h} f(x) \, dx = \text{parabolic area under } f(0) \, f(h) \, f(2h) \]
\[ = \frac{h}{3} \left[ f(0) + 4f(h) + f(2h) \right] \]

Compound Simpson Rule

\[ \int_a^b f(x) \, dx = \frac{h}{3} \left[ f(a) + 4f(a+h) + 2f(a+2h) + \cdots + 4f(b-h) + f(b) \right] \]
\[ = \frac{h}{3} \sum w \times f \quad \quad h = \frac{b - a}{n} \]
Numerical Integration – Example

Use numerical integration with 4 strips to find

\[ \int_{1}^{5} \frac{1}{x} \, dx \]

\[ n = 4 \quad h = \frac{(b-a)}{n} = \frac{(5-1)}{4} = 1 \]

<table>
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<th>w</th>
<th>w*f</th>
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</tbody>
</table>

\[ \Sigma w*f = 3.3667 \quad \Sigma w*f = 4.8667 \]

\[ (h/2)*\Sigma w*f = 1.6833 \quad (h/3)*\Sigma w*f = 1.6222 \]

Compare these estimates with the true value of

\[ \int_{1}^{5} \frac{1}{x} \, dx = \ln 5 = 1.6094 \]
PROPERTIES OF DETERMINANTS

1. $\text{Det } A' = \text{Det } A$

2. If any two rows (or columns) of a matrix are interchanged, then the sign of the determinant is reversed.

3. If a matrix contains a zero row or column, then its determinant is zero.

4. If a row or column of a matrix is multiplied by a scalar, then the value of its determinant is multiplied by that scalar.

5. If any row (or column) of a determinant is a multiple of another row (or column), the value of the determinant is zero.

6. If a multiple of a row (or column) is added to another row (or column), the value of the determinant does not change.

7. If $\text{U}$ is a triangular matrix, $\text{Det } \text{U}$ is the product of its diagonal entries.

Note that properties 6 & 7 allow us to evaluate determinants using Gaussian Elimination techniques.